

# The Szegő Curve and Laguerre polynomials with large negative parameters

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## Abstract

We study the asymptotic zero distribution of the rescaled Laguerre polynomials,  $L_n^{(\alpha_n)}(nz)$ , with the parameter  $\alpha_n$  varying in such a way that  $\lim_{n \rightarrow \infty} \alpha_n/n = -1$ . The connection with the so-called Szegő curve will be showed.

## 1 Introduction

The definition and many properties of the Laguerre polynomials  $L_n^{(\alpha)}$  can be found in Ch. V of Szegő's classic memoir [21]. Given explicitly by

$$L_n^{(\alpha)}(z) = \sum_{k=0}^n \binom{n+\alpha}{n-k} \frac{(-z)^k}{k!}, \quad (1.1)$$

or, equivalently, by the well-known Rodrigues formula

$$L_n^{(\alpha)}(z) = \frac{(-1)^n}{n!} z^{-\alpha} e^z \left( \frac{d}{dz} \right)^n [z^{n+\alpha} e^{-z}], \quad (1.2)$$

they can be considered for arbitrary values of the parameter  $\alpha \in \mathbb{C}$ . In particular, (1.1) shows that each  $L_n^{(\alpha)}$  depends analytically on  $\alpha$  and no degree reduction occurs:  $\deg L_n^{(\alpha)} = n$  for all  $\alpha \in \mathbb{C}$ .

For  $\alpha > -1$  it is well-known the orthogonality of  $L_n^{(\alpha)}(x)$  on  $[0, +\infty)$  with respect to the weight function  $x^\alpha e^{-x}$ ; in particular, all their zeros are simple and belong to  $[0, +\infty)$ . In the general case,  $\alpha \in \mathbb{C}$ ,  $L_n^{(\alpha)}(z)$  may have complex zeros; the only multiple zero can appear at  $z = 0$ , which occurs if and only if  $\alpha \in \{-1, -2, \dots, -n\}$ . In this case we have

$$L_n^{(-k)}(z) = (-z)^k \frac{(n-k)!}{n!} L_{n-k}^{(k)}(z), \quad (1.3)$$

which shows that  $z = 0$  is a zero of multiplicity  $k$  for  $L_n^{(-k)}(z)$ .

In a series of papers ([7], [8] and [12]), asymptotics for rescaled Laguerre polynomials  $L_n^{(\alpha_n)}(nz)$  were analyzed, under the assumption that  $\lim_{n \rightarrow \infty} \alpha_n/n = A \in \mathbb{R}$ . In [12] the authors obtained the weak zero asymptotics for the case where  $A < -1$ , by means of classical (logarithmic) potential theory. To this end, it played a key role a full set of non-hermitian orthogonality relations satisfied by Laguerre polynomials in a class of open contours in  $\mathbb{C}$ . Unfortunately, this analysis could not be extended to the other cases, since for this approach it is essential the connectedness of the complement to the support of the asymptotic distribution of zeros (see e.g. [5] and [18]). However, the authors formulated in [12] a conjecture for the case  $-1 < A < 0$ , which was proved in some cases and refused in others in [8], by means of the Riemann-Hilbert approach (which has been previously used by the same authors in [7] to obtain strong asymptotics in the case  $A < -1$ ). A similar study for Jacobi polynomials with varying nonstandard parameters has been carried out in [9], [11] and [13].

Jacobi or Laguerre polynomials with real parameters (and in general, depending on the degree  $n$ ) appear naturally as polynomial solutions of hypergeometric differential equations, or in the expressions of the wave functions of many classical systems in quantum mechanics (see e.g. [2]).

In [12], the authors also formulated a conjecture for the case  $A = -1$ , but up to now this problem has remained open. Observe that, by (1.3), when  $k = n$  we have:

$$L_n^{(-n)}(z) = (-1)^n \frac{1}{n!} z^n.$$

There is another particular situation corresponding to the case  $A = -1$  which is very well-known in the literature: when  $\alpha_n = -n - 1$ , we have:

$$L_n^{(-n-1)}(z) = (-1)^n \sum_{k=0}^n \frac{z^k}{k!},$$

and thus, in this case the Laguerre polynomials agree (up to a possible sign) with the partial sums of the exponential series. In a seminal paper, G. Szegő [20] showed that the zeros of the rescaled partial sums of the exponential series,  $\sum_{k=0}^n \frac{(nz)^k}{k!} = (-1)^n L_n^{(-n-1)}(nz)$ , approach the so-called the Szegő curve:

$$\Gamma = \{z \in \mathbb{C}, |ze^{1-z}| = 1, |z| \leq 1\}, \quad (1.4)$$

which is a closed curve around the origin passing through  $z = 1$  and crossing once the negative real semiaxis  $(-\infty, 0)$  (see Figure 1). See also [15] for a detailed study of the Szegő curve and some related problems in approximation of functions. Recently, T. Kriecherbauer et al. [6] obtained uniform asymptotic expansions for the partial sums of the exponential series by

means of the Riemann-Hilbert analysis. Also, in [3], the authors studied the asymptotics of orthogonal polynomials with respect to modified Laguerre weights of the type

$$z^{-n+\nu} e^{-Nz} (z-1)^{2b},$$

where  $n, N \rightarrow \infty$  with  $N/n \rightarrow 1$  and  $\nu$  is a fixed number in  $\mathbb{R} \setminus \mathbb{N}$ .

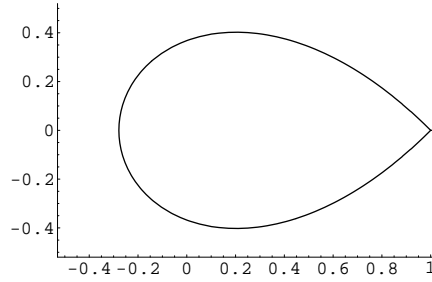


Figure 1: The Szegő curve.

In this paper, the weak zero asymptotics of rescaled Laguerre polynomials  $L_n^{(\alpha_n)}(nz)$ , with  $\lim_{n \rightarrow \infty} \alpha_n/n = -1$  will be analyzed. For it, we will prove that such rescaled Laguerre polynomials are asymptotically extremal on certain well defined curves in the complex plane.

The outline of the paper is as follows. In sect 2, the main result about the weak zero asymptotics of the rescaled Laguerre polynomials is announced, and in sect. 3, some basic facts on potential theory and asymptotically extremal polynomials are recalled. Finally, the proofs are given in sect. 4.

## 2 Main Result

Along with the Szegő curve (1.4), we need to introduce the family of level curves:

$$\Gamma_r = \{z \in \mathbb{C}, |ze^{1-z}| = e^{-r}, |z| \leq 1\}, \quad 0 \leq r < +\infty, \quad (2.1)$$

while for  $r = \infty$  we take  $\Gamma_\infty = \{0\}$ . Observe that  $\Gamma_0 = \Gamma$ , the Szegő curve. We consider the usual counterclockwise orientation. All the level curves  $\Gamma_r$  ( $0 \leq r < +\infty$ ) are closed contours such that  $\{0\} \subset \text{Int}(\Gamma_r)$  and  $\Gamma_{r'} \subset \text{Int}(\Gamma_r)$ , for  $r' > r$ . On the sequel, the interior of  $\Gamma_r$  will be denoted by  $G_r$ . Associated with this family of curves, consider for  $0 \leq r < +\infty$  the family of measures:

$$d\mu_r(z) = \frac{1}{2\pi i} \frac{1-z}{z} dz, \quad z \in \Gamma_r, \quad (2.2)$$

and set  $d\mu_\infty(z) = \delta_0$ .

Let us recall the definition of balayage (or sweeping out) of a measure (see e.g. [16]). Given an open set  $\Omega$  with compact boundary  $\partial\Omega$  and a positive measure  $\sigma$  with compact support in  $\Omega$ , there exists a positive measure  $\hat{\sigma}$ , supported in  $\partial\Omega$ , such that  $\|\sigma\| = \|\hat{\sigma}\|$  and

$$V^{\hat{\sigma}}(z) - V^{\sigma}(z) = \text{const}, \text{ qu.e. } z \notin \Omega, \quad (2.3)$$

where  $\text{const} = 0$  when  $\Omega$  is a bounded set, and a property is said to be satisfied for “quasi-every” (qu.e.)  $z$  in a certain set, if it holds except for a possible subset of zero (logarithmic) capacity. Then,  $\hat{\sigma}$  is said to be the balayage of  $\sigma$  from  $\Omega$  onto  $\partial\Omega$ .

Now, we have the following:

**Lemma 1** *The a priori complex measure (2.2) is a unit positive measure in  $\Gamma_r$  (2.1), for  $0 \leq r < +\infty$ . Moreover,  $\mu_r$  is the balayage of  $\delta_0$  from  $G_r$  onto  $\Gamma_r$ , where  $\delta_0$  denotes the Dirac Delta at  $z = 0$ .*

Now, for each  $n \in \mathbb{N}$ , consider the “pathological” subset of negative integers  $\mathbb{S}_n = \{-n, -(n-1), \dots, -2, -1\}$ . Hereafter, suppose that  $\alpha_n \notin \mathbb{S}_n$ .

Finally, denote by  $\text{dist}(\alpha_n, \mathbb{S}_n) > 0$  the minimal distance between  $\alpha_n$  and the set  $\mathbb{S}_n$ .

**Theorem 1** *Consider a sequence of rescaled Laguerre polynomials  $\{L_n^{(\alpha_n)}(nz)\}_{n \in \mathbb{N}}$ , such that  $\lim_{n \rightarrow \infty} \frac{\alpha_n}{n} = -1$  and*

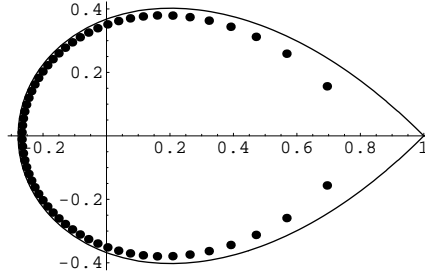
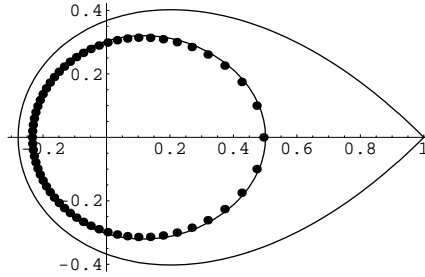
$$\lim_{n \rightarrow \infty} [\text{dist}(\alpha_n, \mathbb{S}_n)]^{1/n} = e^{-r}, \quad (2.4)$$

*for some  $r \geq 0$ . Then, the contracted zeros of Laguerre polynomials asymptotically follow the measure  $d\mu_r$  in (2.2) on the curve  $\Gamma_r$  (2.1). For  $r = +\infty$ , the limit measure is  $d\mu_\infty = \delta_0$ .*

**Remark 1** The results above also hold when dealing with infinite subsequences  $\{L_n^{(\alpha_n)}(nz)\}_{n \in \Lambda}$ ,  $\Lambda \subset \mathbb{N}$ .

**Remark 2** Observe that the case  $r = 0$  in Theorem 1 is generic, because it takes place when parameters  $\alpha_n$  do not approach, or, at least, do not approach exponentially fast, the set of integers  $\mathbb{S}_n$  (see Figure 2). On the other hand, when  $r > 0$  and, so, parameters approach the set of integers  $\mathbb{S}_n$  exponentially fast, the Szegő curve  $\Gamma$  is replaced by a level curve  $\Gamma_r$  which surrounds  $z = 0$  and is strictly contained in the interior of  $\Gamma$  (see Figure 3). Finally, when  $r = \infty$ , i.e., when parameters approach the set  $\mathbb{S}_n$  faster than exponentially, the limit measure reduces to a Dirac mass at  $z = 0$ .

**Remark 3** The weak asymptotics in the case  $A = -1$ , characterized for the set of measures (2.2) and the corresponding set of closed curves (2.1), is the natural matching between the solutions of the cases  $A < -1$  (see [7] and [12]) and  $-1 < A < 0$  (see [8]). For those cases


 Figure 2: The Szegő curve and the zeros of  $L_{60}^{(-60.1)}(60z)$ .

 Figure 3: Zeros of  $L_{60}^{(-60+10^{-5})}(60z)$  and the curve  $\Gamma_r$ , for  $r = \frac{1}{12} \ln 10$ .

the following full set of non-hermitian orthogonality relations for the Laguerre polynomials with parameters  $\alpha \in \mathbb{C}$  was used:

$$\int_{\Sigma} L_n^{(\alpha)}(z) z^k z^{\alpha} e^{-z} dz = 0, \quad k = 0, \dots, n-1,$$

where  $\Sigma$  is any unbounded contour in  $\mathbb{C} \setminus [0, \infty)$ , connecting  $+\infty + iy$  and  $+\infty - iy$ , for some  $y > 0$ , and the branch in  $z^{\alpha}$  is taken with the cut along the positive real axis (see [7, Lemma 2.1]). In [12] this full set of orthogonality relations allowed to apply seminal results by H. Stahl [18] and A. Gonchar and E. A. Rakhmanov [5] on the asymptotic behavior of complex orthogonal polynomials. Indeed, it was proved that zeros of the rescaled Laguerre polynomials accumulate on a closed contour  $C$  in  $\mathbb{C} \setminus [0, \infty)$  which is “symmetric” (in the “Stahl’s sense”, see [17]–[18]) with respect to the external field  $\varphi(z) = \frac{1}{2}(-A \log |z| + \operatorname{Re} z)$ , and that they asymptotically follow the equilibrium distribution on  $\bar{C}$  in presence of the external field  $\varphi$ . In the proof of the main result in this paper, it will be showed that the zeros of the rescaled Laguerre polynomials in the present case also asymptotically follow the equilibrium distribution of  $\Gamma_r$  in presence of the external field  $\varphi$  above (for  $A = -1$ ),  $\Gamma_r$  being a symmetric contour with respect to this external field. That is, although the theorems by H. Stahl and A. Gonchar-E. A. Rakhmanov cannot be applied in this case since the complement to the support is disconnected, the conclusions still hold.

### 3 On asymptotically extremal polynomials

Throughout this section, some topics in potential theory which are needed for the proof of our main result will be recalled. For more details the reader can consult the monography [16].

First, let us precise the notion of admissible weights.

**Definition 1** Given a closed set  $\Sigma \subset \mathbb{C}$ , we say that a function  $\omega : \Sigma \rightarrow [0, \infty)$  is an admissible weight on  $\Sigma$  if the following conditions are satisfied (see [16, Def.I.1.1]):

- (a)  $\omega$  is upper semi-continuous;
- (b) the set  $\{z \in \Sigma : \omega(z) > 0\}$  has positive (logarithmic) capacity;
- (c) if  $\Sigma$  is unbounded, then  $\lim_{|z| \rightarrow \infty, z \in \Sigma} |z|\omega(z) = 0$ .

Given such an admissible weight  $\omega$  in the closed set  $\Sigma$ , and setting  $\varphi(z) = -\log \omega(z)$ , we know (see e.g. [16, Ch.I]) that there exists a unique measure  $\mu_\omega$ , with (compact) support in  $\Sigma$ , for which the infimum of the weighted (logarithmic) energy

$$I_\omega(\mu) = - \int \int \log |z - x| d\mu(z) d\mu(x) + 2 \int \varphi(x) d\mu(x)$$

is attained. Moreover, setting  $F_\omega = I_\omega(\mu_\omega) - \int \varphi d\mu_\omega$ , which is called the modified Robin constant, we have the following property, which uniquely characterizes the extremal measure  $\mu_\omega$ :

$$V^{\mu_\omega}(z) + \varphi(z) \begin{cases} = F_\omega, & \text{qu.e. } z \in \text{supp } \mu_\omega, \\ \geq F_\omega, & \text{qu.e. } z \in \Sigma, \end{cases}$$

where for a measure  $\sigma$ ,  $V^\sigma$  denotes its logarithmic potential, that is,

$$V^\sigma(z) = - \int \log |z - x| d\sigma(x).$$

Now, let  $\Sigma$  be a closed set and  $\omega$  an admissible weight on  $\Sigma$ . Then, a sequence of monic polynomials  $\{p_n\}_{n \in \mathbb{N}}$  is said to be asymptotically extremal with respect to the weight  $\omega$  if it holds (see [16]):

$$\lim_{n \rightarrow \infty} \|\omega^n p_n\|_\Sigma^{1/n} = \exp(-F_\omega), \quad (3.1)$$

where, as usual,  $\|\cdot\|_K$  denotes the sup-norm in the set  $K$ . The study of weighted polynomials of the form  $\omega(z)^n P_n(z)$  has applications to many problems in approximation theory (see e.g. the monographies [16] and [22]). It is well known that if for each  $n \in \mathbb{N}$ ,  $T_n^\omega$  is the  $n$ -th

(weighted) Chebyshev polynomial with respect to the weight  $\omega^n$ , that is, if it is the (unique) monic polynomial of degree  $n$  for which the infimum

$$t_n^\omega = \inf\{\|\omega^n P\|_\Sigma, P(z) = z^n + \dots\}$$

is attained, then the sequence  $\{T_n^\omega\}$  satisfies the asymptotic behavior given in (3.1) (see [16, Ch.III]).

Under mild conditions on the weight  $\omega$ , in [16, Ch.III] it is shown that the zeros of such sequences of polynomials asymptotically follow the equilibrium measure  $\mu_\omega$ , in the sense of the weak-\* convergence. Indeed, we have the following result (see [16, Th.III.4.1] or the previous paper [14]):

**Theorem 2** *Let  $\omega$  be an admissible weight such that the support of the corresponding equilibrium measure  $\mu_\omega$ ,  $S_\omega$ , has zero Lebesgue planar measure. Let  $\{p_n\}_{n \in \mathbb{N}}$  be a sequence of monic polynomials of respective degrees  $n = 1, 2, \dots$  satisfying:*

$$\lim_{n \rightarrow \infty} \|\omega^n p_n\|_{S_\omega}^{1/n} = \exp(-F_\omega), \quad (3.2)$$

where  $F_\omega$  denotes the modified (by the external field  $\varphi = -\ln \omega$ ) Robin constant. Then, the following statements are equivalent:

(a)  $\nu(p_n) \rightarrow \mu_\omega$  in the weak-\* sense, where  $\nu(p_n)$  denotes the unit zero counting measure associated to  $p_n$ , that is,  $d\nu(p_n) = \frac{1}{n} \sum_{p_n(\zeta)=0} \delta_\zeta$ .

(b) For each bounded component  $R$  of  $\mathbb{C} \setminus S_\omega$  and each infinite sequence  $N \subset \mathbb{N}$ , there exist  $z_0 \in R$  and  $N_1 \subset N$  such that

$$\lim_{n \rightarrow \infty, n \in N_1} |p_n(z_0)|^{1/n} = \exp(-V^{\mu_\omega}(z_0)). \quad (3.3)$$

**Remark 4** In [4, Theorem 5], condition (3.2) is replaced by the weaker condition:

$$\limsup_{n \rightarrow \infty} \omega(z) |p_n(z)|^{1/n} \leq \exp(-F_\omega), \text{ qu.e. } z \in S_\omega. \quad (3.4)$$

**Remark 5** It is clear that the balayage of a measure (see (2.3)) is a very particular case of equilibrium measure in an external field. Since Lemma 1 says that measure  $\mu_r$  is the balayage of  $\delta_0$  from  $G_r$  onto its boundary  $\Gamma_r$ , it means that

$$V^{\mu_r}(z) = -\log |z|, \quad z \in \Gamma_r. \quad (3.5)$$

Taking into account the expression of  $\Gamma_r$ , (3.5) implies both

$$V^{\mu_r}(z) + \operatorname{Re} z = r + 1, \quad z \in \Gamma_r, \quad (3.6)$$

and

$$V^{\mu_r}(z) + \varphi(z) = \frac{r+1}{2}, \quad z \in \Gamma_r, \quad (3.7)$$

where

$$\varphi(z) = \frac{1}{2} (\log |z| + \operatorname{Re} z), \quad (3.8)$$

(see Remark 3 above).

For the proof of Theorem 1, taking into account Theorem 2, it will be proved that the rescaled Laguerre polynomials are asymptotically extremal with respect to the weight  $\omega = e^{-\varphi}$  in the compact set given by the closed contour  $\Gamma_r$  (using (3.4)), along with the fact that they satisfy the local behavior (3.3).

## 4 Proofs

### 4.1 Proof of Lemma 1

Take into account that the level curves  $\Gamma_r$ , for  $0 \leq r < \infty$ , given by (2.1) are, in fact, trajectories of the quadratic differential (see e.g. [19])

$$-\frac{(z-1)^2}{z^2} (dz)^2,$$

or, what is the same,  $\Gamma_r$  may be defined in the form:

$$\Gamma_r = \left\{ z \in \mathbb{C} / \operatorname{Re} \int_1^z \left( 1 - \frac{1}{t} \right) dt = r \right\}. \quad (4.1)$$

Expression (4.1) shows that (2.2) is real-valued in  $\Gamma_r$  and does not change its sign. Moreover, by a straightforward application of the Cauchy theorem, we have that

$$\mu_r(\Gamma_r) = \int_{\Gamma_r} d\mu_r(t) = 1.$$

Now, we will prove that  $\mu_r$  is the balayage of  $\delta_0$  from  $G_r = \operatorname{Int}(\Gamma_r)$  onto  $\Gamma_r$ .

To this end, consider the function  $\phi(z) = ze^{1-z}$ . It is easy to see that  $\phi$  conformally maps  $G_r$  onto the disk  $\mathbb{D}_r = \{w \in \mathbb{C} / |w| < r\}$ ,  $0 \leq r < \infty$ , in the  $w$ -plane (see [20] and [15]). Thus, from (2.2), we have:

$$d\mu_r(z) = \frac{1}{2\pi i} \left( \frac{1}{z} - 1 \right) dz = \frac{1}{2\pi i} \frac{\phi'(z)}{\phi(z)} dz = \frac{1}{2\pi i} \frac{dw}{w} = \frac{d\theta}{2\pi},$$

where  $w = re^{i\theta} = \phi(z)$ , and  $z \in \Gamma_r$ . Therefore, (2.2) is the preimage of the normalized arc-length measure on the circle  $\mathbb{T}_r = \partial\mathbb{D}_r$  under the mapping  $w = \phi(z)$ , that is, the harmonic measure at  $z = 0$  with respect to the domain  $G_r$ . But this fact implies that (2.2) is the balayage of  $\delta_0$  from  $G_r$  onto  $\Gamma_r$  (see [10, p. 222]).



## 4.2 Proof of Theorem 1

In Remark 5, it was shown that  $\mu_r$  is also the equilibrium measure in  $\Gamma_r$  in the external field  $\varphi$  (3.8).

Moreover, (3.7) shows that the corresponding modified Robin constant is given by:

$$F_\omega = \frac{r+1}{2}. \quad (4.2)$$

On the other hand, the function  $g(z) = V^{\mu_r}(z) + \operatorname{Re} z$  is harmonic in  $\overline{G_r}$  and, by (3.6),  $g(z) \equiv r+1$ ,  $z \in \Gamma_r$ . Then, it yields that  $g(z) \equiv r+1$ ,  $z \in \overline{G_r}$ . In particular,

$$V^{\mu_r}(0) = r+1. \quad (4.3)$$

From (4.2), in order to prove (3.4) we need to show that

$$\limsup_{n \rightarrow \infty} \omega(z) |p_n(z)|^{1/n} \leq e^{-\frac{r+1}{2}}, \text{ qu.e. } z \in \Gamma_r,$$

for the monic polynomial  $p_n(z) = \widehat{L}_n^{(\alpha_n)}(nz)$  and the weight  $\omega(z) = e^{-\varphi(z)}$ , which taking into account the expression of  $\Gamma_r$ , is equivalent to prove:

$$\limsup_{n \rightarrow \infty} e^{-\operatorname{Re} z} |p_n(z)|^{1/n} \leq e^{-(r+1)}, \text{ qu.e. } z \in \Gamma_r. \quad (4.4)$$

Now, since by (1.1)  $L_n^{(\alpha_n)}(nz) = l_n^{\alpha_n} z^n + \dots$ , with

$$l_n^{\alpha_n} = (-1)^n \frac{n^n}{n!}, \quad (4.5)$$

we have that (4.4) is equivalent to

$$\limsup_{n \rightarrow \infty} e^{-\operatorname{Re} z} |L_n^{(\alpha_n)}(nz)|^{1/n} \leq e^{-r}, \text{ qu.e. } z \in \Gamma_r. \quad (4.6)$$

In addition, we should prove that there exists a point  $z_0 \in G_r$  for which (3.3) is attained. Thus, choosing  $z_0 = 0$ , and taking into account (4.3), it is enough to show that

$$\lim_{n \rightarrow \infty} |p_n(0)|^{1/n} = e^{-(r+1)},$$

or what is the same, by (4.5),

$$\lim_{n \rightarrow \infty} \left| L_n^{(\alpha_n)}(0) \right|^{1/n} = e^{-r}. \quad (4.7)$$

Now, we are going to prove (4.7) and (4.6) under the conditions in Theorem 1.

### 4.2.1 Proof of (4.7)

Take into account that by (1.1),

$$L_n^{(\alpha_n)}(0) = \binom{n + \alpha_n}{n},$$

and let  $h_n \in \{1, 2, \dots, n\}$  be such that

$$\text{dist}(\alpha_n, \mathbb{S}_n) = |\alpha_n + h_n|.$$

Thus, by (2.4), we have:

$$\lim_{n \rightarrow \infty} |\alpha_n + h_n|^{1/n} = e^{-r},$$

and, therefore, to prove (4.7) it should be satisfied:

$$\lim_{n \rightarrow \infty} \left( \frac{|(n + \alpha_n)(n + \alpha_n - 1) \dots (1 + \alpha_n)|}{n! |\alpha_n + h_n|} \right)^{\frac{1}{n}} = 1. \quad (4.8)$$

Let us suppose, first, that  $\alpha_n \geq -n - \frac{1}{2}$ . Then, it is easy to see that

$$|(n + \alpha_n)(n + \alpha_n - 1) \dots (1 + \alpha_n)| = |\alpha_n + h_n| \prod_{k=1}^{n-h_n} |\alpha_n + h_n + k| \prod_{k=1}^{h_n-1} |\alpha_n + h_n - k|,$$

and taking into account that  $\frac{2k-1}{2} \leq |\alpha_n + h_n \pm k| \leq \frac{2k+1}{2}$ , for any integer  $k \geq 1$ , it yields

$$\prod_{k=1}^{h_n-1} \frac{2k-1}{2} \prod_{k=1}^{n-h_n} \frac{2k-1}{2} \leq \frac{|(n + \alpha_n)(n + \alpha_n - 1) \dots (1 + \alpha_n)|}{|\alpha_n + h_n|} \leq \prod_{k=1}^{h_n-1} \frac{2k+1}{2} \prod_{k=1}^{n-h_n} \frac{2k+1}{2}.$$

Thus, denoting  $a_l = \prod_{k=1}^l \frac{2k+1}{2} = \frac{2(l+1)!}{2^{2(l+1)-1}(l+1)!}$ ,  $l \geq 1$ , and  $a_0 = 1, a_{-1} = 2$ . Then,

$$\frac{1}{2^2} a_{h_n-2} a_{n-h_n-1} \leq \frac{|(n + \alpha_n)(n + \alpha_n - 1) \dots (1 + \alpha_n)|}{|\alpha_n + h_n|} \leq a_{h_n-1} a_{n-h_n}, \quad 1 \leq h_n \leq n.$$

On the other hand, if  $\alpha_n < -n - \frac{1}{2}$  (and thus,  $h_n = n$ ),

$$|(n + \alpha_n - 1)(n + \alpha_n - 2) \dots (1 + \alpha_n)| \leq \prod_{k=1}^{n-1} (-\alpha_n - k) = \frac{\Gamma(-\alpha_n)}{\Gamma(-(\alpha_n + n - 1))} = \frac{\Gamma(-\alpha_n)}{\Gamma(\delta_n + 1)},$$

from which it yields

$$a_{n-1} \leq |(n + \alpha_n - 1) \dots (1 + \alpha_n)| \leq \frac{\Gamma(-\alpha_n)}{\Gamma(\delta_n + 1)}.$$

Now, since

$$\lim_{n \rightarrow \infty} \left( \frac{a_{h_n-1-s} a_{n-h_n-s}}{n!} \right)^{\frac{1}{n}} = 1, \quad s = 0, 1,$$

and

$$\lim_{n \rightarrow \infty} \left( \frac{\Gamma(-\alpha_n)}{n! \Gamma(\delta_n + 1)} \right)^{\frac{1}{n}} = 1,$$

then (4.8) follows.

#### 4.2.2 Proof of (4.6)

Let us denote

$$k_n = \min([- \alpha_n], n), \quad \alpha_n = -k_n - \delta_n, \quad \delta_n > 0, \quad (4.9)$$

where, as usual,  $[\cdot]$  denotes the integer part of a real number. It is clear that  $-k_n \in \mathbb{S}_n$  and if  $k_n < n$ , then  $0 < \delta_n < 1$ .

It also holds

$$\text{dist}(\alpha_n, \mathbb{S}_n) = \begin{cases} \delta_n, & \text{if } \alpha_n < -n, \\ \min(\delta_n, 1 - \delta_n), & \text{if } \alpha_n > -n. \end{cases}$$

In order to prove (4.6), the following integral representation will be used (see [1, formula (6.2.22)]):

$$e^{-x} L_n^{(\alpha)}(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_x^\infty (t - x)^{\beta - \alpha - 1} e^{-t} L_n^{(\beta)}(t) dt, \quad (4.10)$$

where  $\beta > \alpha$  and the path of integration is any simple smooth path connecting  $x \in \mathbb{C}$  with  $+\infty$ . Thus, setting  $\beta = -k_n$  and  $\alpha = \alpha_n$  in (4.10) and taking into account (4.9), we have:

$$e^{-x} L_n^{(\alpha_n)}(x) = \frac{1}{\Gamma(\delta_n)} \int_x^\infty (t - x)^{\delta_n - 1} e^{-t} L_n^{(-k_n)}(t) dt,$$

or what is the same, after some calculations,

$$e^{-nx} L_n^{(\alpha_n)}(nx) = \frac{n^{\delta_n}}{\Gamma(\delta_n)} \int_x^\infty (t - x)^{\delta_n - 1} e^{-nt} L_n^{(-k_n)}(nt) dt. \quad (4.11)$$

Now, since  $k_n \in \{1, \dots, n\}$ , making use of (1.3), (4.11) may be written in the form:

$$e^{-nx} L_n^{(\alpha_n)}(nx) = (-1)^{k_n} \frac{n^{\delta_n + k_n} (n - k_n)!}{n! \Gamma(\delta_n)} \int_x^\infty (t - x)^{\delta_n - 1} t^{k_n} e^{-nt} L_{n-k_n}^{(k_n)}(nt) dt. \quad (4.12)$$

On the other hand, taking into account the Rodrigues formula (1.2), (4.12) yields:

$$\begin{aligned} e^{-nx} L_n^{(\alpha_n)}(nx) &= (-1)^n e^{-n} \frac{n^{\delta_n+k_n}}{n! \Gamma(\delta_n)} \int_x^\infty (t-x)^{\delta_n-1} [\phi(t)^n]^{(n-k_n)} dt \\ &= \Delta_n F_n(x), \end{aligned}$$

where, as above,  $\phi(t) = te^{1-t}$  and

$$F_n(x) = \int_x^\infty (t-x)^{\delta_n-1} [\phi(t)^n]^{(n-k_n)} dt.$$

Let us denote by  $x_0 = x_0(r)$  the unique point where the curve  $\Gamma_r$  meets the positive real semiaxis. Now, using the freedom in the choice of the path of integration, it will consist of two arcs: the first goes from  $x$  to  $x_0$  through the curve  $\Gamma_r$  (by the shortest way), and the corresponding integral will be denoted by  $G_n(x)$ ; the second goes from  $x_0$  to  $\infty$  along the positive real semiaxis, and we will denote this integral by  $H_n(x)$ . Thus,  $F_n(x) = G_n(x) + H_n(x)$ .

We are going to estimate  $G_n(x)$ , for  $x \in \Gamma_r \setminus \{x_0\}$ .

Suppose first that  $k_n = n$ , and hence,

$$G_n(x) = \int_x^{x_0} (t-x)^{\delta_n-1} \phi(t)^n dt.$$

For it, consider the natural arc-length parametrization:  $t = t(s)$ , so that  $t(0) = x$  and  $t(s_0) = x_0$ , for some positive real number  $s_0$ . In addition, recall that  $|\phi(t)| = e^{-r}$ , for  $t \in \Gamma_r$ . Since the path of integration is a smooth rectifiable Jordan arc (even for the case when  $r = 0$ , since the path is entirely contained in the upper, or lower, half of  $\Gamma_0 = \Gamma$ ), we have

$$\begin{aligned} |G_n(x)| &= \left| \int_0^{s_0} (t(s) - t(0))^{\delta_n-1} (\phi(t(s)))^n t'(s) ds \right| \\ &\leq \|\phi\|_{\Gamma_r}^n \int_0^{s_0} |t(s) - t(0)|^{\delta_n-1} |t'(s)| ds \\ &\leq e^{-rn} \int_0^{s_0} |t(s) - t(0)|^{\delta_n-1} |t'(s)| ds. \end{aligned} \tag{4.13}$$

Now, take into account that there exist two positive constants  $k, C$ , such that  $k \leq |t'(s)| \leq C$ ,  $s \in [0, s_0]$ , and set  $A_n = \begin{cases} C^{\delta_n}, & \text{if } \delta_n \geq 1, \\ C k^{\delta_n-1}, & \text{if } 0 < \delta_n < 1. \end{cases}$  Then, by classical mean value theorem, (4.13) implies:

$$|G_n(x)| \leq A_n e^{-rn} \frac{s_0^{\delta_n}}{\delta_n}, \tag{4.14}$$

where  $\lim_{n \rightarrow \infty} A_n^{1/n} = 1$ . On the other hand, when  $k_n < n$ , it follows

$$G_n(x) = \int_x^{x_0} (t-x)^{\delta_n-1} [(\phi(t))^n]^{(n-k_n)} dt.$$

Proceeding analogously as above, it holds

$$|G_n(x)| \leq A_n \left\| [\phi^n]^{(n-k_n)} \right\|_{\Gamma_r} \frac{s_0^{\delta_n}}{\delta_n},$$

and thus, by applying the Cauchy integral formula in an arbitrarily small circle around  $t$ , we have for  $t$  in the segment of curve  $\Gamma_r$  connecting  $x$  to  $x_0$ ,

$$\begin{aligned} \left| [\phi(t)^n]^{(n-k_n)} \right| &\leq (n-k_n)! \epsilon^{-n+k_n} e^{\epsilon n} (|\phi(t)| + \epsilon e^2)^n = \\ &= (n-k_n)! \epsilon^{-n+k_n} e^{\epsilon n} (e^{-r} + \epsilon e^2)^n, \end{aligned}$$

for  $\epsilon > 0$  arbitrarily small. Hence,

$$|G_n(x)| \leq A_n (n-k_n)! \epsilon^{-n+k_n} e^{n\epsilon} (e^{-r} + \epsilon e)^n \frac{s_0^{\delta_n}}{\delta_n}, \quad (4.15)$$

for  $\epsilon > 0$ . Since  $\lim_{n \rightarrow \infty} \frac{k_n}{n} = 1$ , we have,

$$\lim_{n \rightarrow \infty} \left( (n-k_n)! \epsilon^{-n+k_n} e^{n\epsilon} (e^{-r} + \epsilon e)^n \right)^{1/n} = e^\epsilon (e^{-r} + \epsilon e), \quad (4.16)$$

for  $\epsilon > 0$ . Observe that taking  $\epsilon \rightarrow 0^+$ , from (4.16), (4.15) agrees with (4.14). Taking into account that  $\lim_{n \rightarrow \infty} \frac{\alpha_n}{n} = -1$ , we have

$$\lim_{n \rightarrow \infty} \left( |\Delta_n| \frac{s_0^{\delta_n}}{\delta_n} \right)^{1/n} = \lim_{n \rightarrow \infty} \frac{n^{\delta_n/n}}{\Gamma(1+\delta_n)^{1/n}} = 1, \quad (4.17)$$

where Stirling formula has been used when  $\delta_n$  is unbounded (recall that  $\delta_n = o(n)$ ). Therefore, by (4.14)-(4.17), it yields

$$\limsup_{n \rightarrow \infty} (|\Delta_n G_n(x)|)^{1/n} \leq e^{-r}, \quad x \in \Gamma_r \setminus \{x_0\}, \quad (4.18)$$

after taking limits when  $\epsilon \rightarrow 0^+$ , if necessary. Note that in this part of the proof (2.4) has not been used.

Now, we are concerned with  $H_n(x)$ . As above, suppose first that  $k_n = n$ , and thus,

$$H_n(x) = \int_{x_0}^{\infty} (t-x)^{\delta_n-1} \phi(t)^n dt,$$

where now the path of integration is contained in the positive real semiaxis. Then, we have

$$H_n(x) = e^n \int_{x_0}^{\infty} \left(1 - \frac{x}{t}\right)^{\delta_n-1} t^{n+\delta_n-1} e^{-(n-1)t} e^{-t} dt. \quad (4.19)$$

Taking into account that there exist two positive constants  $M, N$ , such that  $M \leq \left|1 - \frac{x}{t}\right| \leq N$ ,  $t \in [x_0, \infty)$ , and setting  $B_n = \begin{cases} N^{\delta_n-1}, & \text{if } \delta_n \geq 1, \\ M^{\delta_n-1}, & \text{if } 0 < \delta_n < 1, \end{cases}$  then,

$$|H_n(x)| \leq e^n B_n \|h\|_{[0,+\infty)} \int_{x_0}^{\infty} e^{-t} dt \leq e^n B_n \|h\|_{[0,+\infty)},$$

where  $\lim_{n \rightarrow \infty} B_n^{1/n} = 1$ ,  $h(t) = t^{n+\delta_n-1} e^{-(n-1)t}$  and it is not hard to see that

$$\|h\|_{[0,+\infty)} = h\left(\frac{n+\delta_n-1}{n-1}\right) = \left(\frac{n+\delta_n-1}{n-1}\right)^{n-\delta_n-1} e^{-(n+\delta_n-1)}.$$

Hence,

$$|H_n(x)| \leq B_n \left(\frac{n+\delta_n-1}{n-1}\right)^{n-\delta_n-1} e^{-(\delta_n-1)}.$$

Therefore,

$$\begin{aligned} |\Delta_n H_n(x)| &\leq \frac{e^{-n} n^{n+\delta_n}}{n! \Gamma(1+\delta_n)} B_n \left(\frac{n+\delta_n-1}{n-1}\right)^{n-\delta_n-1} e^{-(\delta_n-1)} \delta_n \\ &\leq C_n \delta_n = C_n \text{dist}(\alpha_n, \mathbb{S}_n), \end{aligned} \quad (4.20)$$

where  $\lim_{n \rightarrow \infty} C_n^{1/n} = 1$ .

On the other hand, when  $k_n < n$ , we have

$$H_n(x) = \int_{x_0}^{\infty} (t-x)^{\delta_n-1} [\phi(t)^n]^{(n-k_n)} dt,$$

and integrating by parts, it yields

$$H_n(x) = (x_0 - x)^{\delta_n-1} [\phi(t)^n]_{t=x_0}^{(n-k_n-1)} + (1 - \delta_n) \int_{x_0}^{\infty} (t-x)^{\delta_n-2} [\phi(t)^n]^{(n-k_n-1)} dt. \quad (4.21)$$

Now, applying again the Cauchy integral formula for  $t \in [x_0, \infty) \subset \mathbb{R}^+$ , it holds:

$$\left| [(\phi(t))^n]^{(l)} \right| \leq l! \epsilon^{-l} e^{2\epsilon n} \phi(t + \epsilon)^n, \quad (4.22)$$

for arbitrarily small  $\epsilon > 0$ .

Then, taking into account (4.21)-(4.22) and setting

$$D_n = (n - k_n - 1)! \epsilon^{-n+k_n+1} e^{2\epsilon n} |x_0 - x|^{\delta_n-1},$$

we have

$$\begin{aligned} |H_n(x)| &\leq D_n \left( \phi(x_0 + \epsilon)^n + (1 - \delta_n) \int_{x_0}^{\infty} |t - x|^{-1} \phi(t + \epsilon)^n dt \right) \\ &\leq D_n \left( \phi(x_0 + \epsilon)^n + (1 - \delta_n) \int_{x_0}^{\infty} \left| 1 - \frac{x - \epsilon}{t} \right|^{-1} t^{-1} \phi(t)^n dt \right). \end{aligned}$$

Finally, we can bound the integral above as in (4.19), which yields

$$|H_n(x)| \leq D_n (\phi(x_0 + \epsilon)^n + (1 - \delta_n) \widetilde{M}^{-1} e^{-1}),$$

where we denote by  $\widetilde{M}$  the lower bound of the function  $|1 - \frac{x - \epsilon}{t}|$ ,  $t \in [x_0, \infty)$ .

Therefore,

$$\begin{aligned} |\Delta_n H_n(x)| &\leq \frac{e^{-n} n^{k_n + \delta_n}}{n! \Gamma(1 + \delta_n)} D_n \left( \phi(x_0 + \epsilon)^n \delta_n + \delta_n (1 - \delta_n) \widetilde{M}^{-1} e^{-1} \right) \\ &\leq R_n \delta_n \phi(x_0 + \epsilon)^n + S_n \delta_n (1 - \delta_n), \end{aligned} \quad (4.23)$$

where  $\lim_{n \rightarrow \infty} R_n^{1/n} = \lim_{n \rightarrow \infty} S_n^{1/n} = 1$ . Taking into account (2.4), we have

$$\lim_{n \rightarrow \infty} [\text{dist}(\alpha_n, \mathbb{S}_n)]^{1/n} = \lim_{n \rightarrow \infty} [\delta_n (1 - \delta_n)]^{1/n} = e^{-r}. \quad (4.24)$$

Now, from (4.20), (4.23) and (4.24), it follows

$$\limsup_{n \rightarrow \infty} (|\Delta_n H_n(x)|)^{1/n} \leq e^{-r}, \quad x \in \Gamma_r \setminus \{x_0\}, \quad (4.25)$$

after taking limits when  $\epsilon \rightarrow 0^+$ , if necessary. Thus, from (4.18) and (4.25), it yields

$$\limsup_{n \rightarrow \infty} (|\Delta_n F_n(x)|)^{1/n} \leq e^{-r}, \quad x \in \Gamma_r \setminus \{x_0\}.$$

It only remains to consider the limit case  $r = \infty$ , which occurs when

$$\lim_{n \rightarrow \infty} [\text{dist}(\alpha_n, \mathbb{S}_n)]^{1/n} = 0.$$

Having in mind the method above, it is not hard to see that in this case, we have that

$$\limsup_{n \rightarrow \infty} e^{-\text{Re } x} |L_n^{(\alpha_n)}(nx)|^{1/n} \leq e^{-s}, \quad x \in \Gamma_s \setminus \{x_0(s)\}, \quad (4.26)$$

for any  $s > 0$ . Thus, applying [4, Theorem 5], (4.26) implies that  $\text{supp } \mu_\infty \subset \overline{G_s}$ , for any  $s > 0$ . Since  $\bigcap_{s>0} \overline{G_s} = \{0\}$ , the conclusion easily follows.

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